Multipole Analysis of Kicks in Collision of Spinning Binary Black Holes

Sarah H. Miller

Center for Relativity and Department of Physics
The University of Texas at Austin, Austin, TX 78712

Richard A. Matzner

Center for Relativity and Department of Physics
The University of Texas at Austin, Austin, TX 78712

ABSTRACT

Thorne and Kidder give expressions which allow for analytical estimates of the "kick", *i.e.* the recoil, produced from asymmetrical gravitational radiation during the interaction of black holes, or in fact any gravitating compact bodies. (The Thorne-Kidder formula uses momentum flux calculations based on the linearized General Relativity of gravitational radiation.) We specifically treat kicks arising in the binary interaction of equal mass black holes, when at least one of the black holes has significant spin, a. Such configurations can produce very large kicks in computational simulations. We consider both *fly-by* and *quasicircular* orbits. For fly-by orbits we find substantial kicks from those Thorne-Kidder terms which are linear in a. For the quasi-circular case, we consider in addition the nonlinear contribution ($O(a^2)$) to the kicks, and provide a dynamical explanation for such terms discovered and displayed by Boyle & Kesden (2007). However, in the cases of maximal kick velocities, the dependence on spin is largely linear (reproduced in numerical results Herrmann et al. (2007a)).

Subject headings: black hole physics — galaxies: nuclei — gravitation — gravitational waves — relativity

1. Introduction

Gravity waves (gravitational radiation) are a product of an extreme gravitational environment; the waves propagate through spacetime itself as fluctuations in the gravitational field (the curvature of spacetime). Experiments, supported by large collaborations such as LIGO (Thorne 1996), now seek to detect these waves of gravity by measuring these subtle fluctuations, further motivating theorists to understand gravity wave emission. Not only is it important to characterize these waves in the context of gravitational wave observations, but also for cosmological models in which black hole mergers play an increasingly important role (di Matteo et al. 2008; Whitaker & van Dokkum 2008).

We consider two interacting black holes in a binary system that radiate away their energy and angular momentum with gravity waves. In addition to the orbital angular momentum of the binary, the individual black holes can also have their own spin angular momenta. If the black holes are gravitationally bound their radiation will eventually lead to inspiral and collapse. The final coalescence of two black holes releases a huge amount of gravitational radiation (up to 10 % of the total rest mass (Washik et al. (2008))). If any sort of asymmetry is present in the binary, *e.g.* if the black holes have unequal masses, or if the black holes have unequal spin angular momenta or spin angular momenta unaligned with their orbital angular momentum, then the asymmetry will be reflected in the gravitational radiation emission of the coalescence, resulting in a "kick" of the final black hole. In this paper we consider the equal mass case of black hole merger where at least one black hole has substantial spin.

Not only can we detect the gravitation radiation of the merger, the kick itself may propel the resultant black hole completely out of the center of a galaxy. Recent numerical simulations of mergers of spinning black holes result in kicks up to 4,000 km/s in quasicircular inspiral (Campanelli et al. 2007) and 10,000km/s in hyperbolic encounters (Healy et al. 2008), which could easily exceed the escape velocity of a galactic nucleus. Shields & Bonning (2008) have presented limits on observing this phenomenon since kicked supermassive black holes may retain a portion of their in-falling matter, which may produce large flares of energy in a characteristic spectrum. Recently, Komossa et al. (2008) have discovered strong observational evidence of a recoiling supermassive black hole with optical emission lines. The black hole appears to have a kick of 2650 km/s!

2. Multipole Formula

Before the modern methods of numerical relativity were developed, Thorne (1980) and Kidder (1995) developed a multipole formula which describes the gravitational radiation kick from dynamical gravitating systems. The formula is based on derivatives of low-order multipoles of the masses and spins of the binary:

$$\frac{dP_i}{dt} = \frac{16}{45}\epsilon_{ijk}I^{(3)}_{jl}H^{(3)}_{kl} + \frac{4}{63}H^{(4)}_{ijk}H^{(3)}_{jk} + \frac{1}{126}\epsilon_{ijk}I^{(4)}_{jlm}H^{(4)}_{klm}. \tag{1}$$

We have included only the terms that depend on the spin. Eq (1) is made up of nth time-derivatives $^{(n)}$ of mass quadrupoles and octupoles, I_{ij} and I_{ijk} respectively, and of spin quadrupoles and octupoles, H_{ij} and H_{ijk} respectively. While the mass quadrupole and octupole are fairly familiar, the spin quadrupoles and octupole are less so. We present formulae below for these quantities, and a scheme to compute the spin multipoles. We use the physical Kerr parameter $a = j/m$, where j is the angular momentum and m is the mass of the black hole, and we point out that H_{ij} and H_{ijk} are linear in the spin a. Thus in Eq (1) the first and third terms are linear in the spin, and the second term is quadratic in the spin.

Because Eq (1) involves time derivatives, it requires knowledge of the motion of the black holes.

But there is no analytical form for two-body motion in General Relativity, so we will use Newtonian physics to describe the motion of the interacting black holes. (One effect of this choice is to confine the motion to the initial orbital plane, $z = 0$.) Thus our results will be accurate but uninteresting for Newtonian motions (slow velocities, large impact parameters or orbital separations). For the interesting case of relativistic interactions (velocities near c, small impact parameters or orbital separations) we will obtain qualitative results, which nonetheless lead to estimates of kick ratios for different configurations and allow an analytical understanding of the kick process. For the quasicircular case our work is complementary to that of Schnittman et al. (2008), which is an extensive study of non-equal mass mergers with either zero or equal but opposite spins; we study *equal mass* mergers with arbitrary ratios and directions of spins on the two black holes.

2.1. Mass Quadrupole and Octupole

The standard expressions for the mass quadrupoles and octupoles are:

$$\begin{aligned} I_{ij} &= \sum m[x_i x_j - \frac{1}{3} r^2 \delta_{ij}], \\ I_{ijk} &= \sum m[x_i x_j x_k - \frac{1}{5} r^2 (\delta_{ij} x_k + \delta_{jk} x_i + \delta_{ki} x_j)], \end{aligned} \tag{2}$$

where the sum is over both black holes. We use Latin letters for spatial indices, x, y, z, and do not distinguish between covariant and contravariant indices. We treat one black hole at a time. The two-hole result is obtained by adding the contribution from each hole. Because in our examples the black holes have equal mass, the mass octupole vanishes on this addition, so that the third term in the formula Eq (1) can be ignored regardless of spin configuration. Since we need the third time derivative of the mass quadrupole, we evaluate $x_k(t)$ and its first three time derivatives.

2.2. Spin Quadrupole and Octupole

The spin multipoles require the baffling concept of spin density, but since we are dealing with Kerr black holes, which have an intrinsic spin dipole moment, we use a trick to evaluate the spin quadrupole. We replace the spin dipole by a fictitious pair of spin charges, of value $q = \pm a$ separated by a distance m, centered at the actual location of the black hole (Herrmann et al. 2007a). This reproduces the dipole angular momentum (am), and allows us to compute the quadrupole directly:

$$\begin{aligned} H_{ij} &= \sum q\ [x_i x_j - \frac{1}{3} r^2 \delta_{ij}] \\ H_{ijk} &= \sum q\ [x_i x_j x_k - \frac{1}{5} r^2 (\delta_{ij} x_k + \delta_{jk} x_i + \delta_{ki} x_j)]\,. \end{aligned} \tag{3}$$

Now the sum is over the two black holes and over the two spin charges for each hole, and the x_i appearing in these formula are offset in the direction of the spin. Strictly one should take a limit

to small separations with the product am held constant, but this is automatic here because the result is proportional to am. We can separately consider the cases for spin components, a_x, a_y, a_z, and add the spin quadrupole or octupole components after individual calculation. (To the lowest order, the spin is parallel-transported along the orbit.)

3. Hyperbolic Fly-by

Consider equal mass black holes approaching each other with impact parameter $2b$ in the x-y plane, and equal and opposite velocities (in the center of mass frame) of v_0. We assume the motion starts along the $x-$axis, and the impact parameter is so large that the angle of deflection is small. Then, rather than explicitly computing the hyperbolic orbit, we can simplify the analysis by assuming that the motion is uniform in the $x-$direction. We will however compute the acceleration (and higher derivatives) using this uniform time dependence. Also, although the radiation reaction can change the orbital plane, we will not consider this feedback in the computation of the radiation. These points should become clear as we work through the analysis. In this section we consider only the lowest, linear-in-a term in Eq (1).

Let us consider only the black hole initially moving along the line $y = +b = const$ with velocity $v = v_0$. It is at $x = 0$ at time $t = 0$. The acceleration is given by the Newtonian result:

$$\frac{d^2x}{dt^2} = -\frac{mx}{4r^3} = -\frac{mv_0t}{4(b^2 + (v_0t)^2)^{\frac{3}{2}}}, \tag{4}$$

where $2r = 2(b^2 + (v_0t)^2)^{\frac{1}{2}}$ is the separation between the black holes. Similarly,

$$\frac{d^2y}{dt^2} = -\frac{mb}{4r^3} = -\frac{mb}{4(b^2 + (v_0t)^2)^{\frac{3}{2}}}. \tag{5}$$

(Our units have the Newtonian constant $G = 1$, and we henceforth take $c = 1$ also.) Integrating Eqs (4) & (5) once in time yields:

$$v_x = v_0 + \frac{m}{4v_0(b^2 + (v_0t)^2)^{\frac{1}{2}}}, \tag{6}$$

and

$$v_y = -\frac{mt}{4b(b^2 + (v_0t)^2)^{\frac{1}{2}}}. \tag{7}$$

Similarly we can differentiate Eqs (6) & (7) to obtain higher time derivatives of the position.

In the Newtonian approximation the motion remains in the $x-y$ plane ($z = 0$), and the motion of the equal mass black holes is symmetrical through the origin. Thus the contribution to the mass quadrupole is equal for the two masses. And, because of the symmetry, only I_{xx}, I_{xy}, I_{yy}, and I_{zz} are in principle nonzero. In particular, $I_{xy} = -\sum mxy$ and $I_{zz} = -\sum \frac{1}{3}m(x^2 + y^2)$.

Note that the deflection angle is of $O(m/bv_0)$. Consistent with our approximation, we keep only the lowest powers of (m/b) in computing multipoles. Our approach is encapsulated in the following rules:

1. write the desired derivative of the multipole in terms of derivatives of x and y;
2. replace undifferentiated x factors by $v_0 t$;
3. replace undifferentiated y factors by b;
4. replace $\dot{x}$ (first time-derivative of x) factors by v_0;
5. in any term with a product of derivatives, first apply rules (3) and (4) above, then drop any term with more than one remaining differentiated factor.

This approach is similar to those of Oohara & Nakamura (1989) and Blanchet et al. (1990).

We demonstrate the approach by evaluating $I^{(3)}_{xy}$ using our prescription. We treat only one hole (the one with $v_x \approx +v_0$):

$$\begin{aligned}
I^{(3)}_{xy} &= m(y\frac{d^3x}{dt^3} + 3\frac{dy}{dt}\frac{d^2x}{dt^2} + 3\frac{d^2y}{dt^2}\frac{dx}{dt} + \frac{d^3y}{dt^3}x) \\
&\to m(b\frac{d^3x}{dt^3} + 3\frac{d^2y}{dt^2}v_0 + \frac{d^3y}{dt^3}v_0 t) \\
&= \frac{m^2 b v_0((v_0 t)^2 - 2b^2)}{2((v_0 t)^2 + b^2)^{\frac{5}{2}}}.
\end{aligned} \tag{8}$$

As another example, we explicitly evaluate $I^{(3)}_{zz}$:

$$\begin{aligned}
I^{(3)}_{zz} &= -\frac{2}{3}m(3v_0\frac{d^2x}{dt^2} + v_0 t\frac{d^3x}{dt^3} + b\frac{d^3y}{dt^3}) \\
&= \frac{m^2 v_0^2 t}{6((v_0 t)^2 + b^2)^{\frac{3}{2}}}.
\end{aligned} \tag{9}$$

Introducing the notation $w = \frac{v_0 t}{b}$, the triply differentiated mass quadrupole for one black hole is:

$$\begin{aligned}
I^{(3)}_{xx} &= -\frac{m^2 v_0}{6b^2}\frac{w(2w^2+11)}{(w^2+1)^{\frac{5}{2}}}, \\
I^{(3)}_{yy} &= \frac{m^2 v_0}{6b^2}\frac{w(w^2+10)}{(w^2+1)^{\frac{5}{2}}}, \\
I^{(3)}_{zz} &= \frac{m^2 v_0}{6b^2}\frac{w}{(w^2+1)^{\frac{3}{2}}}, \\
I^{(3)}_{xy} &= \frac{m^2 v_0}{2b^2}\frac{w^2-2}{(w^2+1)^{\frac{5}{2}}},
\end{aligned} \tag{10}$$

and other components are zero. These mass quadrupoles are for one black hole only, so in work below we include the contribution of the second equal mass black hole, which doubles these moments.

We work out ${}^xH_{xx}$ explicitly to demonstrate the method introduced in Section § 2.2. We assume spin on only *one* black hole. (To indicate the direction of the spin component generating the spin-quadrupole, a_x in this example case, we include a leading label ^{x} on the symbol ${}^xH_{ij}$.) For the particle with velocity $v_x \approx +v_0$, we have:

$$\begin{aligned} {}^xH_{xx} &= a_x((x+\frac{m}{2})^2 - \frac{1}{3}((x+\frac{m}{2})^2+y^2)) \\ &\quad -a_x((x-\frac{m}{2})^2 - \frac{1}{3}((x-\frac{m}{2})^2+y^2)), \\ &= \frac{2a_x}{3}((x+\frac{m}{2})^2 - (x-\frac{m}{2})^2), \\ &= \frac{4a_xmx}{3}. \end{aligned} \tag{11}$$

Similarly,

$$\begin{aligned} {}^xH_{yy} &= a_x(y^2 - \frac{1}{3}((x+\frac{m}{2})^2+y^2)) \\ &\quad -a_x(y^2 - \frac{1}{3}((x-\frac{m}{2})^2+y^2)), \\ &= -\frac{2a_xmx}{3} \\ &= {}^xH_{zz}, \\ {}^xH_{xy} &= a_xmy, \end{aligned} \tag{12}$$

and others zero. (We used $z = 0$.) Also,

$$\begin{aligned} {}^yH_{xx} &= a_x(x^2 - \frac{1}{3}(x^2+(y+\frac{m}{2})^2)) \\ &\quad -a_x(x^2 - \frac{1}{3}((x^2+(y+\frac{m}{2})^2)) \\ &= -\frac{2a_ymy}{3} \\ &= {}^yH_{zz}, \\ {}^yH_{yy} &= \frac{4a_ymy}{3}, \\ {}^yH_{xy} &= a_ymx, \end{aligned} \tag{13}$$

and

$$\begin{aligned} {}^zH_{xz} &= a_zmx, \\ {}^zH_{yz} &= a_zmy, \end{aligned} \tag{14}$$

and others zero.

After differentiating and combining components, we have:

$$H^{(3)}_{xx} = \frac{2m}{3}(2a_x\frac{d^3x}{dt^3} - a_y\frac{d^3y}{dt^3})$$

$$
\begin{aligned}
&= \frac{m^2 v_0(2a_x(2w^2-1)-3a_y w)}{6b^3(w^2+1)^{\frac{5}{2}}}, \\
H^{(3)}_{yy} &= \frac{m^2 v_0(-a_x(2w^2-1)+6a_y w)}{6b^3(w^2+1)^{\frac{5}{2}}}, \\
H^{(3)}_{zz} &= -\frac{m^2 v_0(a_x(2w^2-1)+3a_y w)}{6b^3(w^2+1)^{\frac{5}{2}}}, \\
H^{(3)}_{xy} &= \frac{m^2 v_0(3a_x w+a_y(2w^2-1))}{4b^3(w^2+1)^{\frac{5}{2}}}, \\
H^{(3)}_{xz} &= \frac{m^2 v_0 a_z(2w^2-1)}{4b^3(w^2+1)^{\frac{5}{2}}}, \\
H^{(3)}_{yz} &= \frac{3m^2 v_0 a_z w}{4b^3(w^2+1)^{\frac{5}{2}}}. \qquad (15)
\end{aligned}
$$

The linear term (the first term) in Eq (1) gives, for instance the force dP_x/dt on the binary system:

$$
\frac{dP_x}{dt} = \frac{16}{45}(I^{(3)}_{xy}H^{(3)}_{xz}+(I^{(3)}_{yy}-I^{(3)}_{zz})H^{(3)}_{yz}). \qquad (16)
$$

The force computed is applied to the total mass, so we find the individual black hole velocity by time integrating Eq (16), using $dt = b/v_0$, and dividing the result by $2m$. The velocities are estimated in km/s for $a_x \approx a_z \approx m$, $v \approx 1$, and $\frac{m}{b} \approx \frac{1}{2}$ (closest approach $= 4m$):

$$
\begin{aligned}
P_x &= 2\times(\frac{4}{45}\frac{m^4 v_0^2}{b^5}a_z\frac{b}{v_0}\int_{-\infty}^{\infty}\frac{1}{(w^2+1)^3}dw) \;=\; 2\times\frac{\pi}{30}(\frac{m}{b})^4 a_z v_0, \qquad (17)\\
v_x &\approx 1962\ km/s. \qquad (18)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\frac{dP_y}{dt} &= \frac{16}{45}((I^{(3)}_{zz}-I^{(3)}_{xx})H^{(3)}_{xz}-I^{(3)}_{xy}H^{(3)}_{yz}), \\
P_y &= 2\times(\frac{4}{45}\frac{m^4 v_0^2}{b^5}a_z\frac{b}{v_0}\int_{-\infty}^{\infty}\frac{w}{(w^2+1)^3}dw), \\
&= 0; \qquad (19)\\
\frac{dP_z}{dt} &= \frac{16}{45}(I^{(3)}_{xx}-I^{(3)}_{yy})H^{(3)}_{xy}+I^{(3)}_{xy}(H^{(3)}_{yy}-H^{(3)}_{xx}), \\
P_z &= 2\times(-\frac{2}{45}\frac{m^4 v_0^2}{b^5}\frac{b}{v_0}\int_{-\infty}^{\infty}\frac{a_x(7w^2+4)}{(w^2+1)^4}+\frac{a_y(2w^3+5w)}{(w^2+1)^4}dw) \\
&= 2\times(-\frac{3\pi}{40}(\frac{m}{b})^4 a_x v_0), \qquad (20)\\
v_z &\approx -4415\ km/s. \qquad (21)
\end{aligned}
$$

Odd integrands integrate to zero in our straight-line integration approximation. The accumulated $x-$velocity is the residual CM motion after the encounter, but it is at most of order 10^{-2} of v_0,

so may be unmeasureable. However the P_z estimate of Eq (21) is substantial. Notice that these estimated kicks are for *one* black hole with spin. If both are spinning, the symmetries of the equal mass orbit dictate that $a_i \to (a_1 - a_2)_i$. Hence equal magnitude oppositely directed spin doubles this kick velocity.

At this point we recall the limitations of these calculations, principally that the calculation of the dynamics is Newtonian. Our result is completely consistent and accurate in the Newtonian small-deflection limit, but the estimate Eq (21) is an extravagant extrapolation to $v_0 = c$. In the absence of a General relativistic 2-body simulation, we can make only qualitative adaptations to relativity. One point to notice is that $\frac{m}{b}$ is half the deflection angle in the high-speed Newtonian limit. For a test body moving near $v = c$ past a central mass, in General Relativity the deflection at a given impact parameter and mass is twice the Newtonian result assuming $v = c$. This suggests that we might obtain the result estimated above from motion with twice the impact parameter.

4. Quasi-Circular Inspiral

To contrast our fly-by calculations above, we now calculate the kicks when equal mass black holes ($m_1 = m_2 = m$) are in a circular orbit in the x-y plane. (In fact, the loss of energy means the orbit spirals inword, so is only quasi-circular, but we assume a circular orbit, with the orbital separation an adjustably shrinking quantity to mimic this energy loss.) We choose the first black hole initially (t=0) at position $x = +d$, where the second black hole would be at $x = -d$, with $2d$ as the "circular orbit separation". The third derivatives of the mass quadrupole components for just the first black hole are thus:

$$\begin{aligned} I^{(3)}_{xx} &= 4md^2\omega^3 \sin 2\omega t, \\ I^{(3)}_{yy} &= -4md^2\omega^3 \sin 2\omega t, \\ I^{(3)}_{xy} &= -4md^2\omega^3 \cos 2\omega t. \end{aligned} \tag{22}$$

and other components are zero. The total differentiated mass quadrupole for the two equal-mass system is twice that given in Eqs (22).

The spin multipoles are calculated with the method described in § 2.2 so that for one black hole, the two spin charges per component can be summed using the following coordinates:

For a_x:

$$\begin{aligned} x_1 &= d\cos\omega t + (m/2), \quad & y_1 &= d\sin\omega t, \quad & z_1 &= 0, \\ x_2 &= d\cos\omega t - (m/2), \quad & y_2 &= d\sin\omega t, \quad & z_2 &= 0. \end{aligned} \tag{23}$$

For a_y:

$$\begin{aligned} x_1 &= d\cos\omega t, \quad & y_1 &= d\sin\omega t + (m/2), \quad & z_1 &= 0, \\ x_2 &= d\cos\omega t, \quad & y_2 &= d\sin\omega t - (m/2), \quad & z_2 &= 0. \end{aligned} \tag{24}$$

For a_z:

$$\begin{aligned} x_1 &= d\cos\omega t, & y_1 &= d\sin\omega t, & z_1 &= +(m/2), \\ x_2 &= d\cos\omega t, & y_2 &= d\sin\omega t, & z_2 &= -(m/2). \end{aligned} \tag{25}$$

Again, as in the fly-by case above, we use the fact that the spin is to lowest order parallel transported along the orbit, which is (quasi-) circular here. The non-zero, third derivatives of the spin quadrupoles are thus:

$$\begin{aligned} {}^xH^{(3)}_{xx} &= \frac{4}{3}a_x dm\omega^3\sin\omega t, & {}^xH^{(3)}_{yy} &= -\frac{2}{3}a_x dm\omega^3\sin\omega t, \\ {}^xH^{(3)}_{zz} &= -\frac{2}{3}a_x dm\omega^3\sin\omega t, & {}^xH^{(3)}_{xy} &= -a_x dm\omega^3\cos\omega t, \\ {}^yH^{(3)}_{xx} &= \frac{2}{3}a_y dm\omega^3\cos\omega t, & {}^yH^{(3)}_{yy} &= -\frac{4}{3}a_y dm\omega^3\cos\omega t, \\ {}^yH^{(3)}_{zz} &= \frac{2}{3}a_y dm\omega^3\cos\omega t, & {}^yH^{(3)}_{xy} &= -a_y dm\omega^3\sin\omega t, \\ {}^zH^{(3)}_{xz} &= a_z dm\omega^3\sin\omega t, & {}^zH^{(3)}_{yz} &= -a_z dm\omega^3\cos\omega t, \end{aligned} \tag{26}$$

and the non-zero, fourth derivatives of the spin octupoles, needed only for the second term of Eq (1), are:

$$\begin{aligned} {}^xH^{(4)}_{xxx} &= \frac{72}{5}a_x d^2m\omega^4\cos 2\omega t, & {}^xH^{(4)}_{xyy} &= -\frac{56}{5}a_x d^2m\omega^4\cos 2\omega t, & {}^xH^{(4)}_{xzz} &= -\frac{16}{5}a_x d^2m\omega^4\cos 2\omega t, \\ {}^xH^{(4)}_{xxy} &= \frac{64}{5}a_x d^2m\omega^4\sin 2\omega t, & {}^xH^{(4)}_{yyy} &= -\frac{48}{5}a_x d^2m\omega^4\sin 2\omega t, & {}^xH^{(4)}_{yzz} &= -\frac{16}{5}a_x d^2m\omega^4\sin 2\omega t, \\ {}^yH^{(4)}_{xxx} &= -\frac{48}{5}a_y d^2m\omega^4\sin 2\omega t, & {}^yH^{(4)}_{xyy} &= \frac{64}{5}a_y d^2m\omega^4\sin 2\omega t, & {}^yH^{(4)}_{xzz} &= -\frac{16}{5}a_y d^2m\omega^4\sin 2\omega t, \\ {}^yH^{(4)}_{xxy} &= \frac{56}{5}a_y d^2m\omega^4\cos 2\omega t, & {}^yH^{(4)}_{yyy} &= -\frac{72}{5}a_y d^2m\omega^4\cos 2\omega t, & {}^yH^{(4)}_{yzz} &= \frac{16}{5}a_y d^2m\omega^4\cos 2\omega t, \\ {}^zH^{(4)}_{xyz} &= 8a_z d^2m\omega^4\sin 2\omega t, & {}^zH^{(4)}_{xxz} &= 8a_z d^2m\omega^4\cos 2\omega t, & {}^zH^{(4)}_{yyz} &= -8a_z d^2m\omega^4\cos 2\omega t. \end{aligned} \tag{27}$$

Note that if both black holes have spin, ${}^cH^{(3)}_{ab}$ is computed by subtracting a similar formula for the second spin: ${}^cH^{(3)}_{ab} \propto (a_{c_1} - a_{c_2})$; and ${}^cH^{(4)}_{abf} \propto (a_{c_1} + a_{c_2})$.

4.1. First Term

We now calculate the first term from Eq (1),

$$\left.\frac{dP_i}{dt}\right._{1^{st}} = \frac{16}{45}\epsilon_{ijk}I^{(3)}_{jl}H^{(3)}_{kl}\,. \tag{28}$$

1^{st} term only:

$$\left.\frac{dP_x}{dt}\right._{1^{st}} = \frac{16}{45}(I^{(3)}_{xy}H^{(3)}_{xz} + I^{(3)}_{yy}H^{(3)}_{yz}),$$

$$
\begin{aligned}
&= 2\times(\frac{64}{45}d^3m^2\omega^6 a_z\sin\omega t),\\
\frac{dP_y}{dt}_{1^{st}} &= \frac{16}{45}(-I^{(3)}_{xx}H^{(3)}_{xz} - I^{(3)}_{xy}H^{(3)}_{yz})\\
&= 2\times(-\frac{64}{45}d^3m^2\omega^6 a_z\cos\omega t),\\
\frac{dP_z}{dt}_{1^{st}} &= \frac{16}{45}(I^{(3)}_{xx} - I^{(3)}_{yy})H^{(3)}_{xy} + I^{(3)}_{xy}(H^{(3)}_{yy} - H^{(3)}_{xx})\\
&= 2\times(\frac{128}{45}d^3m^2\omega^6(a_y\cos\omega t - a_x\sin\omega t)). \qquad (29)
\end{aligned}
$$

The first term of Eq (1) is linear in spin, consistent with computational simulations as seen in Herrmann et al. (2007a,b). The " $2\times$ " accounts for the two black holes of the system (doubling the mass quadrupole of a single black hole).

For arbitrary orientation of spin of magnitude a, the components are simply $a_x = a\sin\theta\cos\varphi$, $a_y = a\sin\theta\sin\varphi$ and $a_z = a\cos\theta$. Above, the a_i are the spin components of just the one black hole that is spinning, but if both holes were spinning, we replace a with $(a_1 - a_2)$.

The circular orbit case presented here is based on Newtonian orbits, which specify frequency as a function of the Newtonian separation:

$$
\omega = \sqrt{\frac{m}{d^3}}. \qquad (30)
$$

4.2. Second Term

With the symmetries of Herrmann et al. (2007a,b), the second, nonlinear term in Eq (1) vanishes identically. However if the spins are not equal in magnitude or not anti-aligned, this nonlinear term does not vanish, implying a quadratic contribution to kick velocity.

We calculate the second, quadratic, term from Eq (1),

$$
\frac{dP_i}{dt}_{2^{nd}} = \frac{4}{63}H^{(4)}_{ijk}H^{(3)}_{jk}\,. \qquad (31)
$$

2^{nd} term only:

$$
\begin{aligned}
\frac{dP_x}{dt}_{2^{nd}} &= \frac{4}{63}(H^{(4)}_{xxx}H^{(3)}_{xx} + H^{(4)}_{xxy}H^{(3)}_{xy} + H^{(4)}_{xyy}H^{(3)}_{yy} + H^{(4)}_{xzz}H^{(3)}_{zz} + H^{(4)}_{xxz}H^{(3)}_{xz} + H^{(4)}_{xyz}H^{(3)}_{yz})\\
&= \frac{16}{315}d^3m^2\omega^7[-\sin\omega t(26a_x^2 + 23a_y^2 + 10a_z^2) - \sin 3\omega t(10a_x^2 - 9a_y^2)\\
&\quad + \ a_xa_y(3\cos\omega t + 11\cos 3\omega t)],\\
\frac{dP_y}{dt}_{2^{nd}} &= \frac{4}{63}(H^{(4)}_{xxy}H^{(3)}_{xx} + H^{(4)}_{xyy}H^{(3)}_{xy} + H^{(4)}_{yyy}H^{(3)}_{yy} + H^{(4)}_{yzz}H^{(3)}_{zz} + H^{(4)}_{yyz}H^{(3)}_{yz} + H^{(4)}_{xyz}H^{(3)}_{xz})
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{16}{315} d^3 m^2 \omega^7 [\cos\omega t(23a_x^2 + 26a_y^2 + 10a_z^2) + \cos 3\omega t(-9a_x^2 + 10a_y^2) \\
&\quad + \ a_x a_y(-3\sin\omega t + 11\sin 3\omega t)], \\
\frac{dP_z}{dt}_{2^{nd}} &= \frac{4}{63}(H^{(4)}_{xzz}H^{(3)}_{xz} + H^{(4)}_{yzz}H^{(3)}_{yz} + H^{(4)}_{xxz}H^{(3)}_{xx} + H^{(4)}_{yyz}H^{(3)}_{yy} + H^{(4)}_{xyz}H^{(3)}_{xy}) \\
&= \frac{16}{315} d^3 m^2 \omega^7 a_z (11a_y \cos\omega t + 5a_y \cos 3\omega t - 11a_x \sin\omega t + 5a_x \sin 3\omega t).
\end{aligned} \tag{32}
$$

The total force of the kick-components (first and second terms of Eq (1) together) can be compared in Fig (1) for the case where only one of two equal mass black holes is spinning at $a = 0.6m$ with a separation of $6m$. Notice that when the spin is perpendicular to the plane ($\theta = 0$), the z-component of the kick vanishes. When the spin is in the plane ($\theta = \frac{\pi}{2}$), the linear term predicts only a $z-$ component to the kick. In fact the in-plane kick components (x & y) are nonzero due to the non-linear contribution found in the second term of the multipole analysis formula (Eq (1)).

With any choice of the orbital phase and in any spin configuration the largest kicks are those linear in spin. The contributions quadratic in spin become comparable to the linear terms only for $a \sim 1$. However we are interested in the general solutions rather than simply the largest kicks, and there exists a range of spin angles where the quadratic behavior of the second term in Eq (1) can be seen to dominate over the first, linear term in some components . This occurs when the spin angle θ satisfies $|\theta - \frac{\pi}{2}| < \frac{\pi}{12}$, that is, when the x-y spin components (orbital plane) are much larger than the z spin component. In this configuration the the x-y component of the kick is dominated by the quadratic contribution, while the z-component of the kick is more than an order or magnitude larger.

Without an analytical 2-body solution, there is substantial ambiguity in converting this to a relativistic specification. An even more serious problem arises from the kick formulae predicting forces (dP_i/dt) whose vertical component oscillates and whose in-plane components rotate with the orbit. If the orbit were strictly circular, the average of the kick would be zero, though the system would execute periodic motion due to the asymmetric radiation. In fact the orbit is only quasi-circular, and shrinks slowly due to gravitational radiation. Eventually the holes either spiral until they disappear behind a common black hole horizon, or enter a final plunge to the horizon. This sudden cutting off means the net kick can be modelled by considering the “last” (quasi-) circular fractional orbit. Such a concept is ambiguous at best because no analytic prescription describes the motion. However, numerical experiments do show kicks, and we can extract their dependence on orbital parameters. It is of great utility give simple analytical predictions, and thus we produce net, effective kicks depending on the phase of the orbit as it finally merges, and parametrized by the fraction of the “last” orbit that contributes the net kick.

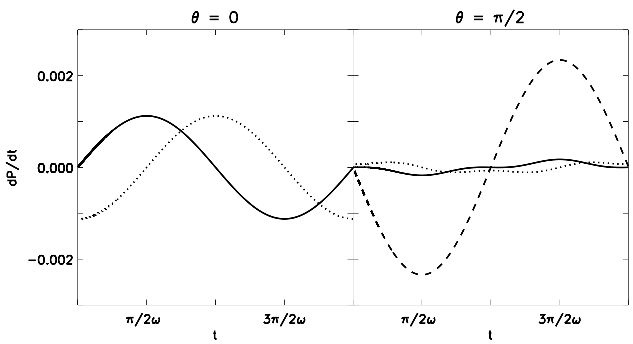

Fig. 1.— Quasi-circular kick-components when the spins ($a \sim 0.6m$) are perpendicular to the plane ($\theta = 0$) and in the plane ($\theta = \frac{\pi}{2}$). The x-component of the kick is the solid line, the y-component is the dotted line, and the z-component is the dashed line. Notice the quadratic effects appearing when the spin is in the plane. Quadratic effects will become comparable to the linear effects only when $a \sim 1$.

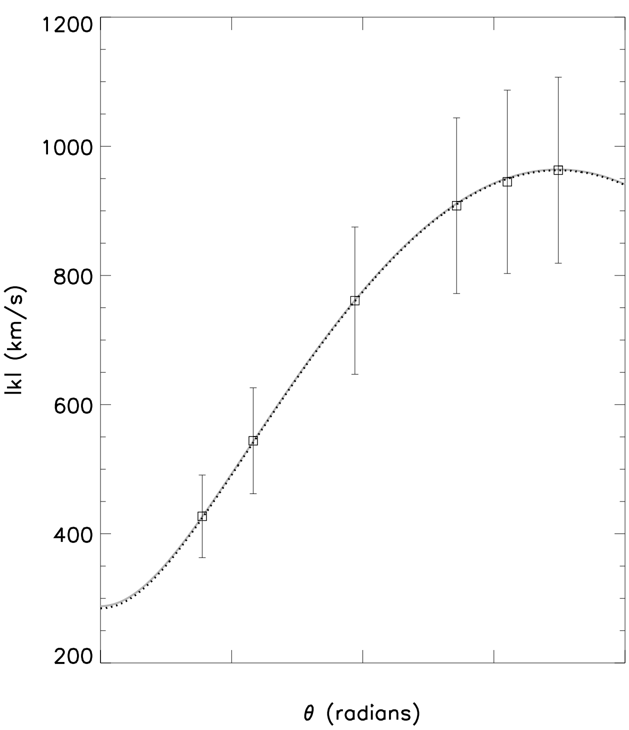

Fig. 2.— Comparing computational result for kick velocities (squares - the Herrmann et al. (2007b) computational B-Series) as a function of spin, to two analytic expressions: Dotted line - Boyle & Kesden (2007) expansion; solid grey line - multipole analysis (this work). Because of our ambiguity in the "last orbit" radius, and in the phase of the fraction of the "last orbit" determining the kick, we fit two parameters: the maximum kick, and the phase.

5. Discussion

It is of interest to compare the kicks from the quasicircular orbits, to the ones from the hyperbolic flyby. For instance, Eq(20) for hyperbolic flyby has a kick velocity

$$v_z \sim \frac{1}{4}(\frac{m}{b})^4(\frac{a_x}{m})v_0. \tag{33}$$

To compare this to the circular orbit result, consider the force, Eq.(29):

$$\frac{dP_z}{dt} \sim 5d^3m^2\omega^6 a_x \sim 5m^2 a_x \frac{m^5}{d^6}, \tag{34}$$

where we used Eq(30). The kick velocity is approximated by multiplying $\frac{dP_z}{dt}$ by a fraction $(\epsilon/2\pi)$ of an orbital period, and dividing by $2m$. Thus:

$$v_z \sim \epsilon\frac{5}{2}(\frac{m}{d})^4(\frac{a_x}{m})(\frac{m}{d^2})\sqrt{\frac{d^3}{m}} \sim \epsilon\frac{5}{2^{\frac{3}{2}}}(\frac{m}{d})^4(\frac{a_x}{m})v_{orbit}. \tag{35}$$

The process "multiplying $\frac{dP_z}{dt}$ by a fraction of an orbital period" summarizes integrating the force for the relevant period at the final plunge, since circular orbits do not produce a net kick.

Equations(33) and (35) are very similar, differing (aside from numerical factors) by v_0 in the hyperbolic case being replaced by ϵv_{orbit}, and the impact parameter by the orbital radius. The quantity ϵ is poorly defined, but is likely less than unity. Also, while v_0 can in principle be very close to unity, the orbital velocity for a given orbital radius will be much less than the flyby velocity with an impact parameter equal to that orbital radius. This suggests that hyperbolic-orbit kicks can in principle be larger (much larger) than quasicircular kicks. Numerical studies (Healy et al. (2008)) confirm these large kicks predicted by the multipole analysis for the fly-by case.

Recently, Boyle & Kesden (2007) presented a spin expansion in order to understand final quantities of binary black hole mergers, such as mass, kick velocity, and spin vector. They consider two Kerr black holes in quasicircular orbits and Taylor expand some final quantity in terms of the spins. Symmetry arguments remove excessive independent terms at each order. Boyle & Kesden (2007) discover second and third order spin contributions that lie beyond the empirical fitting formulas which come from post-Newtonian, linear dependence fits from simulations. We compare our results to their expansion for a numerical black hole binary simulation in Herrmann et al. (2007b), referred to as the "B-series" (§IV C of Boyle & Kesden (2007)). In this particular case, equal mass black holes have oppositely directed equal-magnitude spins ($a = 0.6m$) lying in the x-z plane (θ_1 =[0, π] while θ_2 =[π, 2π] for BH_1 and BH_2, respectively). Fig (2) compares the Herrmann et al. numerically computed points with the Boyle & Kesden expansion and our multipole analysis for the resultant kick magnitude. Because of our ambiguity in the "last orbit" radius, and in the phase of the fraction of the "last orbit" determining the kick, we fit two parameters: the maximum kick, and the phase, and we plot the sum of Eq(29) and Eq(32). We find a tight agreement between these three methods, and the quadratic contribution in Boyle & Kesden expansion can be quantitatively understood with the quadrupole and octupole of the binary using the multipole formula.

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